Finding Bounds on Ehrhart Quasi-Polynomials

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ABSTRACT

Numerous problems in program analysis, can be reduced to finding bounds on the number of integer points in a convex set, or the solution of a more general polyhedral counting problem. For a large class of applications the solution of such a counting problem can be expressed as a piecewise Ehrhart quasi-polynomial in the parameters.

This work presents methods to find bounds on quasi-polynomials over discrete domains, starting from bounds on polynomials over continuous domains.

KEYWORDS: Ehrhart quasi-polynomials, discrete domain, polynomials, extrema

1 What are (Ehrhart) quasi-polynomials?

The general form of a(n) (Ehrhart) quasi-polynomial is defined in terms of periodic numbers.

Definition 1 *A* rational periodic number $u(n)$ *is a function* $\mathbb{Z} \to \mathbb{Q}$*, such that* $u(n) = u(n')$ *whenever* $n \equiv n' \pmod{d}$*, where* $d \in \mathbb{N}$ *is called the period of* $u(n)$*.*

A periodic number can be represented by an array of rational numbers:

$$u(n) = [u_0, u_1, \ldots, u_{d-1}]_n \ ,$$

which means

$$u(n) = u_i \text{ if } n \equiv i \pmod{d} \ .$$

Alternatively, $u(n)$ can be expressed using fractional parts of linear expressions. For example,

$$\left\{\frac{n}{3}\right\} + \left\{\frac{n}{2}\right\} = \frac{1}{6}[0, 5, 4, 3, 2, 7]_n \ ,$$

where the notation $\{.\}$ denotes the fractional part, i.e. $\{x\} = x - \lfloor x \rfloor$.

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[2]Harald Devos is supported by I.W.T. grant 060068

Definition 2 *A* quasi-polynomial $f(p)$ *of degree* g *in one variable* p *is a polynomial expression in* p *over the rational periodic numbers, i.e.*

$$f(p) = \sum_{i=0}^{g} u_i(p)p^i,$$

where the $u_i(p)$ *are periodic numbers.*

The definitions of periodic number and quasi-polynomial can be extended to the multivariate case [Verd07b]. A *piecewise* quasi-polynomial is a function that is a quasi-polynomial in each element of a partition of the domain into polyhedral subdomains, called *chambers*.

2 Where do they come from?

Our interest in quasi-polynomials stems from the following theorem.

Theorem 1 (Clauss [Clau98]) *The number of integer points in a parametric polytope* P_p *of dimension* n *is expressed on each chamber of a partition of the parameter space by a quasi-polynomial of degree* n *in* p*.*

Here p represents an array of integer parameters, $p \in \mathbb{Z}^m$. Also more general counting problems, defined with linear inequalities and the logical operators $\wedge$, $\vee$, $\forall$, $\exists$, $\neg$ (so-called Presburger formula), have a piecewise quasi-polynomial as solution [Verd07b]. In the experiments, the `barvinok` library [Verd07a], which implements and extends the techniques proposed by Barvinok [Barv94], is used to solve the counting problems.

3 Why do we need bounds on quasi-polynomials?

Many problems in program analysis can be reduced to finding bounds on the solution of a polyhedral counting problem of the form defined in Section 2. For example, if the number of live elements in a program at a certain point of execution is expressed as a (piecewise) quasi-polynomial in the parameters and the point of execution, then the memory usage is the maximum of that (piecewise) quasi-polynomial over all points of execution.

4 How do we find bounds?

Quasi-polynomials are only defined over a discrete domain. Therefore, we will first look at the difference between the extrema of a polynomial over a continuous domain and the extrema in the corresponding discrete subdomain (the integer points in the domain).

4.1 Continuous versus discrete domain extrema of polynomials

We will use the notation $\overline{F}_{\rm c}$ and $\underline{F}_{\rm c}$ for the maximum and minimum of a polynomial in the continuous domain $\mathcal{D}$, and $\overline{F}_{\rm d}$ and $\underline{F}_{\rm d}$ for the extrema in the discrete subdomain $\mathbb{Z}^n \cap \mathcal{D}$. If one uses the continuous extrema as an approximation of the discrete extrema, the approximation error on the range, RE, is defined as

$$RE = \begin{cases} \dfrac{(\overline{F}_{\mathrm{c}} - \underline{F}_{\mathrm{c}}) - (\overline{F}_{\mathrm{d}} - \underline{F}_{\mathrm{d}})}{\overline{F}_{\mathrm{d}} - \underline{F}_{\mathrm{d}}} & \text{when } \overline{F}_{\mathrm{d}} \neq \underline{F}_{\mathrm{d}} \text{ ;} \\ 0 & \text{when } \overline{F}_{\mathrm{d}} = \underline{F}_{\mathrm{d}} \wedge \overline{F}_{\mathrm{c}} = \underline{F}_{\mathrm{c}} \text{ ;} \\ \infty & \text{when } \overline{F}_{\mathrm{d}} = \underline{F}_{\mathrm{d}} \wedge \overline{F}_{\mathrm{c}} \neq \underline{F}_{\mathrm{c}} \text{ .} \end{cases} \tag{1}$$

The maximal possible RE for an arbitrary polynomial depends on the degree of the polynomial and the size of the domain. It becomes smaller for larger domains or lower degrees. A detailed study is found in [Devo]. In many cases, the continuous domain extrema can be used as an approximation of the discrete domain extrema. In other cases, partially evaluating the polynomial may improve the accuracy. For example, the polynomial

$$f(x, y) = -2x^2y^2 + 6x^2y - 2x^2 + 6xy^2 - 18xy + 6x - 2y^2 + 6y \tag{2}$$

over the domain $[0, 9] \times [0, 2]$ has (continuous) extrema 139.5 and -108, while the extrema in the discrete subdomain $\mathbb{Z}^2 \cap [0, 9] \times [0, 2]$ are 112 and -108. Partially evaluating $f(x, y)$ for the possible values of y leads to 3 polynomials in one variable

$$\begin{aligned} f_0(x) &= f(x, 0) = -2x^2 + 6x \\ f_1(x) &= f(x, 1) = 2x^2 - 6x + 4 \\ f_2(x) &= f(x, 2) = 2x^2 - 6x + 4, \end{aligned}$$

with continuous domain extrema $9/2$, -108, and twice 112 and $-1/2$, respectively. The extrema in continuous and discrete domain now coincide.

4.2 Converting quasi-polynomials into polynomials

To apply the results of Section 4.1 on quasi-polynomials they have to be converted to polynomials. This can be done by eliminating periodic numbers by introducing new variables. For example,

$$f(n) = n^2 \left\{ \frac{n-2}{100} \right\} + 2n \qquad 0 \leq n \leq 10000,$$

can be converted to

$$g(n, q) = n^2 \frac{q}{100} + 2n \qquad 0 \leq n \leq 10000, 0 \leq q \leq 99.$$

Note that only the points of the discrete subdomain of g for which $q = (n-2) \bmod 100$ correspond to discrete points of the domain of f.

Partial evaluation of the new variables leads to different methods, with varying tradeoffs between accuracy and computation cost. A detailed overview is given in [Devo]. The methods were implemented using the `barvinok` library [Verd07a].

5 Experiments

The techniques have been used to estimate the memory usage of sets of schedules of a 1-D FIR (Finite Impulse Response) filter and matrix multiplication. Bernstein expansion is used

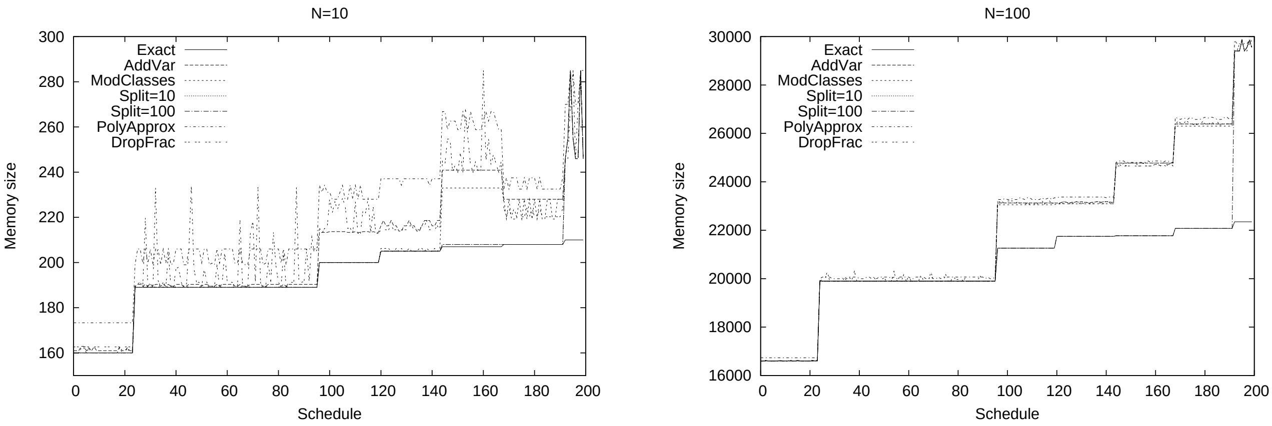

Figure 1: Comparison of different methods to estimate the memory usage of a matrix multiplication as a function of the schedule (sorted along exact usage). N is the size of the matrices.

to derive bounds on polynomials over continuous domains. The results indicate that different methods perform better on different classes of quasi-polynomials or subdomains. Probably, a hybrid method applying different methods in each chamber for each quasi-polynomial could combine the best of all methods. Therefore, a heuristic will be needed that selects the most appropriate method based on the properties of the quasi-polynomial and the chamber.

Acknowledgments

The authors would like to thank Sven Verdoolaege for fruitful discussions and implementing part of the techniques in the `barvinok` library.

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