A Stochastic Model for the Interconnection Topology of Digital Circuits

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Abstract— **Rent's rule has been successfully applied to a priori estimation of wire length distributions. However, this approach is very restrictive: the circuits are assumed to be homogeneous. In this brief, recursive clustering is described as a more advanced model for the partitioning behavior of digital circuits. It is applied to predict the variance of the terminal count distribution. First, the impact of the block degree distribution is analyzed with a simple model. A more refined model incorporates the effect of stochastic self-similarity. Finally, the model is further extended to describe the effects of heterogeneity. This model is a promising candidate for more accurate a priori estimation tools.**

Keywords—**Interconnection topology, recursive clustering, stochastic model, Rent's rule, a priori estimation.**

I. Introduction

THE COMPLEXITY of the interconnection topology of a digital circuit is well captured by Rent's rule [1] and the corresponding Rent exponent. This empirical law states that, when a circuit is partitioned into a set of more or less equally sized modules subject to a certain terminal minimization objective, there exists a relationship between the average terminal count $\overline{T}$ and the module size $\overline{B}$:

$$\overline{T} = k\overline{B}^p \tag{1}$$

The parameters k and p are known as the *Rent coefficient* and the *Rent exponent*, respectively. Rent's rule has been applied numerous times, especially in the field of a priori wire length estimation [2], [3], [4]. All these techniques use Rent's rule as a simple model for the partitioning behavior of logic circuits. The circuits are assumed to be hierarchically homogeneous (absence of Rent region II [1]) and spatially homogeneous in a very strict sense: for any partition all modules are assumed to have an identical number of terminals. Rent's rule is used in a deterministic sense, which is clearly a very restrictive interpretation.

In reality the terminal counts for a set of modules in a partition show a certain spread. Fig. 1 shows the partitioning behavior and Rent fit for ISPD98 [5] circuit `ibm02`. The partitions were obtained by recursively bipartitioning the circuit with hMetis [6]. Rent's rule describes the sample mean of these terminal counts, which is only a first-order statistic. In this brief we apply *recursive clustering* as a more realistic model to describe the partitioning behavior of circuits. This model essentially captures the stochastic properties of self-similarity in circuit graphs. Therefore, it allows the estimation of higher-order statistics such as the sample variance of the terminal count distribution.

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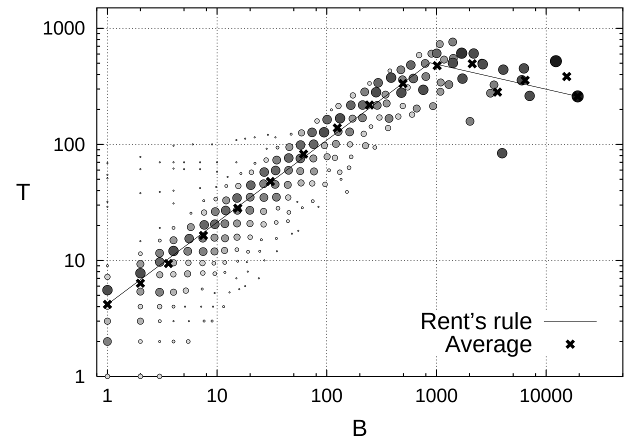

Fig. 1. Terminal count distribution and average terminal count for different partitions of `ibm02`. The size of the circles corresponds to the percentage of modules that have T terminals and B blocks.

II. Recursive clustering

A k-way partitioning can be obtained directly or by a sequence of multiple bipartitionings, where each module is again partitioned until the k modules are obtained. The latter is often referred to as a *hierarchical* or *recursive* partitioning scheme. For theoretical derivations it is often useful to assume that the circuit has exactly 2^H blocks and that every bipartitioning is perfectly balanced. After i bipartitioning phases one obtains a partition of 2^i modules of size 2^{H-i} blocks, which will be referred to as the partition at hierarchical level $h = H - i$.

The recursive partitioning model has been applied many times in the field of wire length estimation. The reverse process, the *recursive clustering* of modules, was first used by Stroobandt et al. [7] for the theoretical derivation of the net degree distribution of homogeneous circuits. One starts from a set of 2^H modules each consisting of a single block at hierarchical level 0. The clustering at level 0 results in 2^{H-1} modules of size 2. After h clustering phases 2^{H-h} modules of size 2^h are generated. Finally, after H clustering phases the complete circuit is created.

Consider the combination of two modules of identical size B with terminal counts T_1 and T_2 into a new module with $B_c = 2B$ blocks and T_c terminals. A number of nets will be connected, and thus a few terminals will be eliminated. The terminal count reduction factor β is defined as

$$\beta = \frac{T_c}{T_1 + T_2} \tag{2}$$

Assume that the circuit is strictly homogeneous and that Rent's rule can be applied to each module individually. Hence $T_1 = T_2 = kB^p$ and $T_c = k(2B)^p$, such that $\beta = 2^{p-1}$. A fixed β and identical building blocks result in a deterministic model where at each hierarchical level all modules have identical terminal counts. A more realistic model can be obtained by relaxing these assumptions: the blocks may have different terminal counts, such that the resulting modules to be combined can have different terminal counts as well. This can be modeled by assuming that the terminal counts $\mathbf{T}_h$ are random variables with identical distributions. When these variables are independent the resulting circuit will be homogeneous in a stochastic sense. Furthermore, β does not need to be fixed but can be a random variable as well. When the stochastic parameters of all these distributions are fixed throughout the recursive clustering, the generated circuit will be hierarchically homogeneous. The terminal count at hierarchical level h is modeled as a random variable given by

$$\mathbf{T}_h = \beta\left(\mathbf{T}_{h-1,1} + \mathbf{T}_{h-1,2}\right) \tag{3}$$

The recursive clustering process represents a model for the partitioning behavior of a circuit. This model will be suitable when the terminal counts for the different modules are representative for the actual values of a typical partitioning of a typical circuit. However, real circuits rarely have exactly 2^H blocks, and often a partitioning scheme that allows a certain imbalance of the modules is applied. Module imbalance can be translated to increased variance of the terminal count distributions, though this effect is usually negligible for real partitionings. When substituting the hierarchy level to the module size ($h = log_2(B)$), theoretical results can be applied to general circuits of any size.

III. Variance of the terminal count

Rent's rule is an estimate for the sample mean of the terminal count distribution for a certain partition. Since this is a first-order statistic, it cannot be applied to more accurate higher-order prediction methods. The spread of the terminal count distribution is very prominent, as can be derived from fig. 1. Visually, the amount of spread seems to remain constant or even diminish for increasing block size due to the compressing effect of the logarithmic scale. In reality the amount of spread, which can be characterized by the sample variance of the terminal count distribution, increases for increasing module size.

In this section we will estimate the sample variance s_T^2 by using the recursive clustering model. In this model the terminal count is a random variable, and the terminal count variance $\sigma_{\mathbf{T}}^2$ is a good estimate for the sample variance of the actual terminal count distribution. First, we will model the influence of the block degree distribution. Then we will refine this model by including the stochastic nature of self-similarity. A further refinement will be obtained by explicitly incorporating heterogeneity as well.

A. Impact of the block degree distribution

A circuit usually consists of different building blocks, with different port counts. At the lowest level of the hierarchy, where each module consists of a single block, the terminal count of each module corresponds to the number of ports of the encapsulated block. The latter is known as the *block degree*. The terminal counts at the lowest level ($B = 1$) can be modeled by random variables $\mathbf{t}_i$ with distribution according to the block degree distribution, which can be derived directly from the circuit netlist. The presence of spread on the block degree will cause a spread of the terminal count distributions at higher levels of the hierarchy as well. To analyze the impact of the block degree distribution, we will neglect the stochastic properties of the self-similarity by assuming β to be constant and given by

$$\beta = 2^{p-1} \tag{4}$$

The recurrence relationship (3) can be expanded as

$$\mathbf{T}_h = \beta^h(\mathbf{t}_1 + \mathbf{t}_2 + \cdots + \mathbf{t}_{2^h}) \tag{5}$$

Assume the random variables $\mathbf{t}_i$ to be mutually independent with identical distributions, which is equivalent to the assumption that the circuit is spatially homogeneous. The expected value and the variance of the terminal count distribution at hierarchical level h is then:

$$\mathbf{E}[\mathbf{T}_h] = \beta^h 2^h \mu_{\mathbf{t}} = 2^{hp}\mu_{\mathbf{t}} \tag{6}$$

$$\mathrm{Var}[\mathbf{T}_h] = \beta^{2h} 2^h \sigma_{\mathbf{t}}^2 = 2^{(2p-1)h}\sigma_{\mathbf{t}}^2 \tag{7}$$

The mean and variance of $\mathbf{t}_i$ can be estimated by the sample mean $\overline{t}$ and sample variance s_t^2 of the block degree distribution. Substituting for the module size $B = 2^h$ we get the following estimates for the mean and variance of the terminal count distribution:

$$\overline{T} = \overline{t}B^p \tag{8}$$

$$s_T^2 = s_t^2 B^{(2p-1)} \tag{9}$$

The first expression reduces to Rent's rule (1) with $k = \overline{t}$. Similar to Rent's rule, the expression for the variance is again a power law relationship. When the Rent exponent $p > 0.5$ the variance increases. For $p = 0.5$ the variance remains constant, and for $p < 0.5$ it decreases. Fig. 2 shows actual and predicted variance (model 1) for `ibm02`, which has a Rent exponent of 0.701. The existence of a power law relationship is confirmed, but the measured exponent does not comply with (9). Of course the measurement of the Rent exponent is not exact, but in this case a Rent exponent of 1.114 would be required to obtain a fairly good match. This is not reasonable, since a Rent exponent can never exceed 1. This indicates that the block degree distribution is not the main cause for the spread of the terminal count distribution at higher hierarchical levels.

B. Stochastic self-similarity

During the recursive partitioning process the factor β can be measured directly. As expected, this factor is not fixed, but varies around a certain average value, which is more or less constant over a large range of hierarchical levels. This is due to the stochastic nature of self-similarity. It can be modeled by considering β as a sequence of random variables

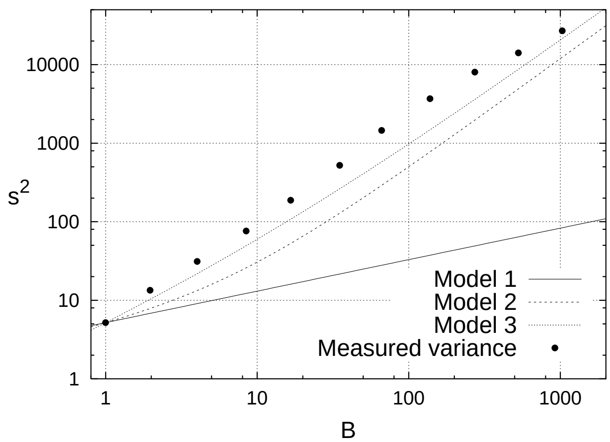

Fig. 2. Measured and predicted variance for circuit `ibm02`.

with certain mean μ_β and variance σ_β^2. As a result, the variance of the terminal count distribution will increase much faster.

The random variables β, $\mathbf{T}_1$ and $\mathbf{T}_2$ in expression (3) are assumed to be mutually independent. The expected value of a product of two mutually independent variables $\mathbf{X}$ and $\mathbf{Y}$ is equal to the product of the expected values, and the variance is given by

$$\mathrm{Var}[\mathbf{XY}] = \mathrm{Var}[\mathbf{X}]\mathrm{Var}[\mathbf{Y}] + \mathrm{Var}[\mathbf{X}]\mathbf{E}[\mathbf{Y}]^2 + \mathrm{Var}[\mathbf{Y}]\mathbf{E}[\mathbf{X}]^2$$

Hence the expected value and the variance of the terminal count distribution becomes:

$$\mathbf{E}[\mathbf{T}_h] = 2\mu_\beta \mathbf{E}[\mathbf{T}_{h-1}] \tag{10}$$

$$\begin{aligned}\mathrm{Var}[\mathbf{T}_h] &= 2\sigma_\beta^2 \mathrm{Var}[\mathbf{T}_{h-1}] + 4\sigma_\beta^2 \mathbf{E}[\mathbf{T}_{h-1}]^2 \\ &\quad + 2\mathrm{Var}[\mathbf{T}_{h-1}]\mu_\beta^2 \\ &= 2(\sigma_\beta^2 + \mu_\beta^2)\mathrm{Var}[\mathbf{T}_{h-1}] \\ &\quad + \left(\frac{\mathbf{E}[\mathbf{T}_h]}{\mu_\beta}\right)^2 \sigma_\beta^2\end{aligned} \tag{11}$$

The solution of recurrence relationship (10) is straightforward:

$$\mathbf{E}[\mathbf{T}_h] = 2^h \mu_\beta^h \mathbf{E}[\mathbf{T}_0] = (2\mu_\beta)^h \mu_{\mathbf{t}} \tag{12}$$

This allows expression (11) to be simplified as:

$$\begin{aligned}\mathrm{Var}[\mathbf{T}_h] &= 2(\sigma_\beta^2 + \mu_\beta^2)\mathrm{Var}[\mathbf{T}_{h-1}] \\ &\quad + \frac{\mu_\beta^2}{\sigma_\beta^2}\mu_{\mathbf{t}}^2 (4\mu_\beta^2)^h\end{aligned} \tag{13}$$

The latter is a recurrence relationship of the form

$$f_h = a\, f_{h-1} + b\, c^h \tag{14}$$

with a known explicit solution:

$$f_h = f_0\, a^h + \frac{bc}{c-a}\left(c^h - a^h\right) \tag{15}$$

In this case,

$$a = 2(\mu_\beta^2 + \sigma_\beta^2) \tag{16}$$

$$b = \frac{\sigma_\beta^2}{\mu_\beta^2}\mu_{\mathbf{t}}^2 \tag{17}$$

$$c = 4\mu_\beta^2 \tag{18}$$

such that

$$bc = 4\sigma_\beta^2 \mu_{\mathbf{t}}^2 \tag{19}$$

$$c - a = 2(\mu_\beta^2 - \sigma_\beta^2) \tag{20}$$

therefore, the solution of (13) is:

$$\begin{aligned}\mathrm{Var}[\mathbf{T}_h] &= \sigma_{\mathbf{t}}^2 2^h (\mu_\beta^2 + \sigma_\beta^2)^h \\ &\quad + \frac{2\sigma_\beta^2 \mu_{\mathbf{t}}^2}{\mu_\beta^2 - \sigma_\beta^2}\left((4\mu_\beta^2)^h - 2^h(\mu_\beta^2 + \sigma_\beta^2)^h\right)\end{aligned} \tag{21}$$

Equation (12) is again Rent's rule. When the measured Rent exponent is p the mean value μ_β can be estimated as

$$\mu_\beta = 2^{p-1} \tag{22}$$

When estimating the mean $\mu_{\mathbf{t}}$ and variance $\sigma_{\mathbf{t}}^2$ of $\mathbf{t}$ by the sample mean $\overline{t}$ and sample variance s_t^2, and the variance σ_β^2 of β by the sample variance s_β^2, expression (12) reduces to (8), and the estimate of the sample variance of the terminal count distribution becomes

$$\begin{aligned}s_T^2 &= \left(s_t^2 - \frac{2s_\beta^2}{2^{2p-2} - s_\beta^2}\overline{t}^2\right) B^{\log_2(2^{2p-1} + 2s_\beta^2)} \\ &\quad + \frac{2s_\beta^2}{2^{2p-2} - s_\beta^2}\overline{t}^2 B^{2p}\end{aligned} \tag{23}$$

The variance consists of two terms. Since $\beta \in [0,1]$ and usually $p > 0.4$ the mean value $\overline{\beta} = 2^{p-1} > 0.66$, such that the sample variance s_β^2 will always be significantly smaller than 2^{2p-2}. Therefore the exponent of the first term will be approximately $2p-1$, such that the second term will dominate for sufficiently large module size.

In fig. 2 the estimated variance based on stochastic self-similarity (model 2) for circuit `ibm02` is displayed as a dashed line. Though still an underestimate, the predicted variance is a much better estimate than the previous model (without stochastic self-similarity). Because of the recursive nature of the model, errors are being accumulated. It appears that still quite a large error is introduced at the lowest levels of the hierarchy. This is due to the existence of heterogeneity, which so far has not been incorporated in the model.

C. Heterogeneity

When bipartitioning a circuit the two parts will usually have a different terminal count. If these subcircuits were to be homogeneous, the average terminal counts would be on straight lines in a double logarithmic plot. Since both lines intersect the T-axis at more or less the same point (somewhere near the average block degree $\overline{t}$), this would imply that the Rent exponents and thus the complexity of both subcircuits are different (fig. 3 left).

These differences in complexity are a manifestation of heterogeneity. If the heterogeneity occurs only at one spe-

$$s_T^2 = \left(s_t^2 - \frac{2 s_\beta^2}{(1-r_T)2^{2p-2} - (1+r_T)s_\beta^2} \bar{t}^2 \right) B^{\log_2[(1+r_T)(2^{2p-1}+2s_\beta^2)]} + \frac{2 s_\beta^2}{(1-r_T)2^{2p-2} - (1+r_T)s_\beta^2} \bar{t}^2 B^{2p} \qquad (25)$$

cific hierarchical level, all subcircuits at that level are by themselves homogeneous circuits. This can easily be incorporated into the recursive clustering model, since the heterogeneity determines a partitioning of the blocks into distinct classes. When clustering the different blocks and resulting modules, only those blocks and modules that consist of the same class can be combined. Once the level of hierarchy is reached where the heterogeneity exists, the class restriction is relaxed and any pair of modules can be combined. The resulting terminal count distributions are a mixture of distributions that correspond to homogeneous circuits.

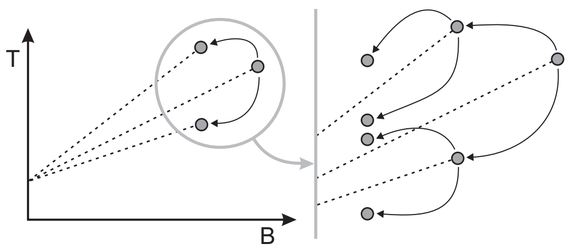

Fig. 3. Heterogeneity in Rent plots.

However, the heterogeneity does usually not occur at one specific level, but exists at all levels of the hierarchy. As a result, the terminal counts of the modules that are combined during the recursive clustering process will be correlated (fig. 3 right). The amount of heterogeneity is characterized by the correlation coefficient $\rho_\mathbf{T}$. In general, the variance of the sum of two correlated random variables with identical distribution is given by

$$\mathrm{Var}[\mathbf{X}_1 + \mathbf{X}_2] = 2\sigma_\mathbf{X}^2 (1+\rho)$$

Hence, the recurrence relationship for the variance becomes:

$$\mathrm{Var}[\mathbf{T}_h] = 2(1+\rho_\mathbf{T})(\sigma_\beta^2 + \mu_\beta^2)\mathrm{Var}[\mathbf{T}_{h-1}] + \frac{\mu_\beta^2}{\sigma_\beta^2} \mu_\mathbf{t}^2 (4\mu_\beta^2)^h \qquad (24)$$

The calculations of the mean and variance are almost identical to those in the previous section. When estimating the correlation coefficient $\rho_\mathbf{T}$ by the sample correlation coefficient r_T, the final estimate for the variance of the terminal count distribution is given by expression (25) at the top of this page.

The estimated variance in region I for circuit `ibm02` is shown in fig. 2 as a dotted line (model 3). An underestimate still exists for the lower hierarchical levels. Due to the accumulation of errors this underestimate appears at the higher hierarchical levels as well. Similar deviations to the measured values have been observed for the other ISPD98 circuits. Based on the Rent characteristics in fig. 1, `ibm02` seems hierarchically homogeneous up to the beginning of the second region. But closer inspection reveals that the Rent exponent is a little higher at the lower levels of the hierarchy, and decreases as the modules grow. These hierarchical inhomogeneities become even more apparent when inspecting the other variables, such as the sample variance s_β^2 and correlation coefficient r_T. The combination of these effects results in an underestimate of the variance increase at lower levels, and an overestimate at higher levels of the hierarchy.

IV. Conclusion

Rent's rule states a relationship between the average terminal count and the module size of a partitioned circuit. For wire length estimation this rule is interpreted in a very strict sense: the circuits are assumed to be hierarchically and spatially homogeneous. As a first-order estimate, all modules are assumed to have the exact number of terminals as predicted by Rent's rule. This deterministic interpretation of Rent's rule is a very restrictive approximation.

Recursive clustering can be used to model the partitioning behavior of digital circuits. Because the stochastic properties of self-similarity in digital circuits are adequately captured, recursive clustering appears to be a valuable candidate for the next generation of a priori estimation tools. We applied recursive clustering to model the terminal count distribution. First, a simple model was derived that only takes the block degree distribution into account. Then, the model was refined to incorporate the effects of stochastic self-similarity. Finally, the model was further extended to describe the effects of heterogeneity more accurately. The model was validated by predicting the variance of the terminal count distribution for all ISPD98 benchmark circuits. Some deviations were observed due to hierarchical heterogeneity.

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