Efficient Representation of Interconnection Length Distributions Using Generating Polynomials*

Dirk Stroobandt†
ELIS Department, Ghent University
Sint-Pietersnieuwstraat 41
B-9000 Gent, Belgium
dstr@elis.rug.ac.be

Herwig Van Marck
Centre for Evolvable Language Engineering,
Flanders Language Valley
St.-Krispijnstraat 7, B-8700 Ieper, Belgium
Herwig.VanMarck@flv.be

ABSTRACT

A renewed interest in interconnection length distribution models leaves many researchers with the task of enumerating shortest distances between cells in a physical architecture. This enumeration process is cumbersome and time-consuming. In this paper, we simplify it by representing interconnection length distributions by generating polynomials. We show that this representation greatly facilitates the enumeration, allows the early calculation of key distribution parameters and provides a very compact representation. We use the inherent symmetry in physical architectures to construct generating polynomials with the composition and convolution techniques. It is shown that the construction of the generating polynomials using these techniques is much simpler than the construction of the distribution itself. We also present an efficient way to extract the final distribution from its generating polynomial.

Categories and Subject Descriptors

G.2.1 [**Mathematics of Computing**]: Discrete Mathematics—*Generating functions, Counting problems*; I.1.1 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Representations*; G.3 [**Mathematics of Computing**]: Probability and Statistics—*Distribution functions*; J.6 [**Computer Applications**]: Computer-aided Engineering—*CAD*; B.8.2 [**Hardware**]: Performance and Reliability—*Performance Analysis and Design Aids*

General Terms

Generating polynomials, Interconnect length distributions, Enumeration, VLSI CAD.

*Supported by an Inter University Attraction Poles research program on optical computing of the Belgian Government.

†Dirk Stroobandt is a Postdoctoral Fellow of the Fund for Scientific Research (F.W.O.) – Flanders, Belgium.

1. INTRODUCTION

Since the advent of deep submicron (DSM) design, on chip interconnections have been the focus of attention. Models of interconnection characteristics are being published regularly. One aspect of dealing with the increasing design problems, is the a priori estimation of design related parameters such as the interconnection length and using the estimates to improve the layout process or make an assessment of the outcome before the process itself is performed. Although such estimates have been initiated as far back as the 1970's with the introduction of Rent's rule [10] and the work on interconnection length estimates by Donath [5], further research on this issue remained unexplored for a long time. It took until the end of the 1990's before it began to have some impact with new interconnection length distribution models [15, 14], especially the highly referenced model by Davis et al [4], and the founding of the workshop on System-Level Interconnect Prediction (SLIP) [13] in 1999.

Interconnection length estimations are now being used to predict interconnect area, number of layers, and average interconnection length in several estimation systems (SUSPENS [1], RIPE [7], GENESYS [6], BACPAC [18], GTX [2]). Research on length estimations has also been introduced in the optical community [11, 21, 19].

Most contributions on interconnection length estimates use a factorization of the interconnection length distribution into two parts [3]: (i) an *occupancy probability* which represents the probability that a given path is being used (occupied) and which is largely independent of the specific system details and (ii) a *site function* which represents the distribution of the number of cell pairs in a physical architecture with respect to the distance between the cells (length of the shortest path connecting both cells). The calculation of the site function requires a tedious enumeration of all possible paths a connection can follow in the system. The enumeration involves convolutions and different support intervals.

In this paper, we present a technique that greatly facilitates the enumeration process by representing interconnection site functions as *generating polynomials* [16]. This allows us to compactly represent an interconnection site function in a single expression. It also facilitates the calculations and allows easy computation of characteristic parameters of the site function (such as the average interconnection length).

Section 2 describes the enumeration problem and introduces generating polynomials as a more suited representation for length distributions. Two techniques for the construction

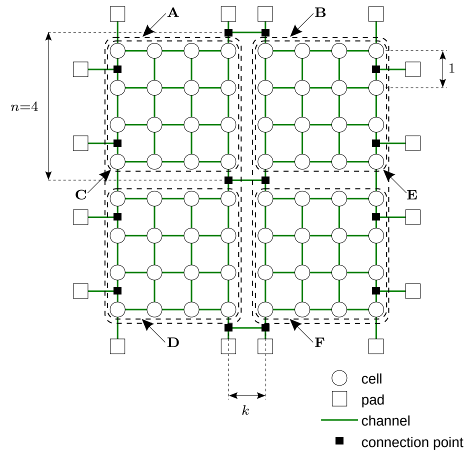

Figure 1: An anisotropic architecture for which we enumerate all shortest paths between cells in A and cells in B.

of the polynomials are highlighted in section 3: *composition* and *convolution*. They both heavily rely on the inherent symmetry in physical architectures. In section 4, the extraction of the resulting length distribution from its generating polynomial representation is described and two examples in section 5 show this process in detail. The appendix contains the proof of a lemma that is used for the extraction.

2. ENUMERATION OF SITE FUNCTIONS

2.1 Problem formulation

Although most research on interconnection length distributions uses fairly simple architectures (two- or three-dimensional isotropic architectures), where there is always a *channel* (possible site for a path) between two neighboring cells, it is very useful to study anisotropic architectures [21, 19], such as the one in figure 1 (loosely connected supercells of highly connected cells). Especially for such more complicated architectures, our technique will prove very beneficial. The site function describes, for each length ℓ, the number of pairs between all cells of a set **A** and all cells of a set **B**, a distance ℓ apart. In the most general case, calculating the site function requires a very specific enumeration. For each pair of cells (one cell from set **A**, one from **B**) the distance between the cells has to be determined (along the shortest path between the cells). Then all such paths are grouped per length and this results in the site function.
The calculation effort can be reduced by (i) representing the distribution function more efficiently and (ii) using the symmetry in the interconnection topology of the architecture.

2.2 Generating polynomials

The site function is a discrete distribution $f(\ell)$ ($\ell \in \mathbb{N}$). For our finite physical architectures, there always exists a maximum value ℓ_{max} for which $f(\ell) = 0$ if $\ell > \ell_{max}$. In this subsection, we introduce an efficient representation of such distributions by using their *generating polynomial*

$$\mathcal{V}(x) = \sum_{\ell=0}^{\infty} f(\ell)\, x^{\ell}. \tag{1}$$

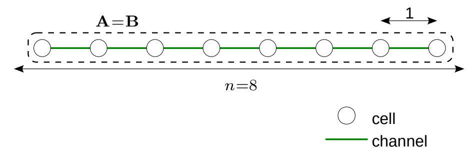

Figure 2: Line of n cells connected to itself.

Of course, one recognizes in equation 1 the moment-generating polynomial function of the distribution $f(\ell)$ and the relationship to Z-transforms [22]. If $f(\ell)$ is a length distribution of a set of paths $\mathcal{P}$, then each path $p \in \mathcal{P}$ is represented in the generating polynomial $\mathcal{V}(x)$ by a term $x^{\ell(p)}$, with $\ell(p)$ the length of the path p. Hence, we can also calculate the generating polynomial by summation over all paths p:

$$\mathcal{V}(x) = \sum_{p \in \mathcal{P}} x^{\ell(p)}. \tag{2}$$

Consider, as an example, the set of (shortest) paths between the cells from the set **A** and the cells from the set **B** in the physical architecture of figure 2 (here, **A** and **B** are equal). The corresponding generating polynomial $\mathcal{V}_1(x)$ can be calculated by summation over all pairs of cells (a, b) with $a \in \mathbf{A}$ and $b \in \mathbf{B}$. Numbering the cells from 0 to $n-1$, the distance between cell i and cell j is given by $|i-j|$ and we obtain

$$\mathcal{V}_1(x) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} x^{|i-j|} = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} x^{(i-j)} + \sum_{j=i+1}^{n-1} x^{(j-i)}\right). \tag{3}$$

Repeatedly using the expression for a geometric series

$$\sum_{i=0}^{n-1} a^k = \frac{a^n - 1}{a-1} \tag{4}$$

equation 3 results in

$$\mathcal{V}_1(x) = \frac{2\,x^{n+1} - n\,x^2 - 2\,x + n}{(x-1)^2}. \tag{5}$$

Although $\mathcal{V}_1(x)$ is represented as a rational function, it is a polynomial because the denominator is a divisor polynomial of the numerator. However, it is beneficial to use the rational function representation.
The benefits of using generating polynomials can already be observed. Some relevant characteristic properties of the distribution can be easily obtained from it. The total number of paths in the distribution is found as

$$\sum_{\ell=0}^{\ell_{max}} f(\ell) = \mathcal{V}(1). \tag{6}$$

One can verify that in the example of figure 2 the total number of paths equals n^2 (the limit in equation 7 is necessary to take care of the singularity of the rational function in $x = 1$ and the rule of de L' Hôpital is used twice to solve it)

$$\begin{aligned}\mathcal{V}_1(1) &= \lim_{x\to 1} \frac{2\,x^{n+1} - n\,x^2 - 2\,x + n}{(x-1)^2} \\ &= \left.\frac{2\,(n+1)\,n\,x^{n-1} - 2\,n}{2}\right|_{x=1} = n^2.\end{aligned} \tag{7}$$

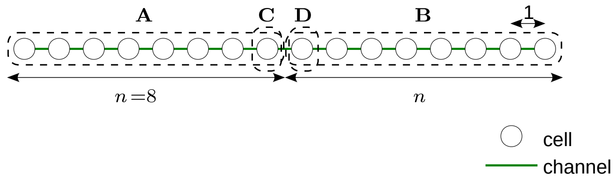

Figure 3: Line of n cells connected to adjacent line of n cells.

Generating polynomials also allow an easy calculation of the average value and all higher order moments. The average value of the distribution, e.g., is found as

$$\frac{\sum_{\ell=0}^{\ell_{max}} \ell\, f(\ell)}{\sum_{\ell=0}^{\ell_{max}} f(\ell)} = \left(\frac{d\mathcal{V}(x)}{dx} \frac{1}{\mathcal{V}(x)} \right) \Bigg|_{x=1} . \tag{8}$$

For our example, we can calculate

$$\left(\frac{d\mathcal{V}_1(x)}{dx} \frac{1}{\mathcal{V}_1(x)} \right) \Bigg|_{x=1} = \frac{(n-1)\,(n+1)}{3\,n}. \tag{9}$$

A second benefit of using generating polynomials is that they allow a compact representation of the distribution function. The generating polynomial for the distribution of lengths between sets **A** and **B** in the physical architecture of figure 3, e.g., is found to be (number cells from right to left in **A** and from left to right in **B**)

$$\mathcal{V}_2(x) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} x^{(i+j+1)} = \frac{x\,(x^n-1)^2}{(x-1)^2}. \tag{10}$$

The distribution $f(\ell)$ corresponding to this generating polynomial[1]

$$f(\ell) = \begin{cases} \ell & (0 \le \ell \le n) \\ 2\,n - \ell & (n \le \ell \le 2\,n) \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

contains several distinct regions for ℓ (depending on a parameter n!). These regions are implicitly present in the generating polynomial. For more complex distributions, it is very beneficial to use the generating polynomial for all calculations so that the distinct regions do not complicate the calculations and only appear at the very last step in which we revert to the representation as in equation 11. The examples in section 5 will make this clear.

3. CONSTRUCTION OF POLYNOMIALS

Consider the sets **A** and **B** of cells and the set **K** of all pairs of cells from **A** and **B**: $\mathbf{K} = \mathbf{A} \times \mathbf{B}$. For each pair of cells from **K**, we need to know the distance between the cells, i.e., the length of the shortest path between them. To efficiently enumerate all such paths, we employ the symmetry inherently present in the interconnection topology of the physical architectures. Two basic techniques are presented, *composition* and *convolution*.

3.1 Composition

The symmetry in the physical architecture can be used to compose the set **K** of cell pairs from a number of less complex sets. For this, we can use the union of sets and the difference of sets. With the representation as generating polynomials this corresponds to the summation or subtraction of generating polynomials. Consider the example of figure 4. To obtain the horizontal component of the path lengths between cells of **A** and cells of **B**, the set of cell pairs **K** can be composed of two other sets $\mathbf{K}'$ and $\mathbf{K}''$, where $\mathbf{K}'$ contains all pairs of all cells in **A** and all cells in **A** and **B** together and $\mathbf{K}''$ contains all pairs of all cells within **A**

$$\mathbf{K}' = \mathbf{A} \times (\mathbf{A} \cup \mathbf{B}) \quad \text{and} \quad \mathbf{K}'' = \mathbf{A} \times \mathbf{A}. \tag{12}$$

The sets **K**, $\mathbf{K}'$ and $\mathbf{K}''$ and their corresponding generating polynomials then relate as

$$\mathbf{K} = \mathbf{K}' \setminus \mathbf{K}'' \quad \text{and} \quad \mathcal{V}_3(x) = \mathcal{V}_3'(x) - \mathcal{V}_3''(x) \tag{13}$$

The generating polynomial $\mathcal{V}_3''(x)$ is calculated in equation 5 as $\mathcal{V}_1(x)$ (figure 2). For the generating polynomial $\mathcal{V}_3'(x)$ (corresponding to $\mathbf{K}'$) we note that the physical architecture (for the horizontal component of the length) is equivalent to a ring of cells. Each cell will hence see the same length distribution (represented by $\mathcal{V}_{3,1}'(x)$) and we can write

$$\mathbf{K}' = \bigcup_{i=0}^{n-1} \mathbf{K}_1(i) \quad \text{and} \quad \mathcal{V}_3'(x) = n\,\mathcal{V}_{3,1}'(x), \tag{14}$$

with $\mathbf{K}_1(i)$ the set of cell pairs with the first cell being cell $a_i \in \mathbf{A}$

$$\mathbf{K}_1(i) = \{a_i\} \times (\mathbf{A} \cup \mathbf{B})\,. \tag{15}$$

In a ring of length $2\,n$, each cell has one cell at distance 0, one cell at distance n and 2 cells at distances in between 0 and n, hence

$$\mathcal{V}_{3,1}'(x) = 1 + x^n + \sum_{i=1}^{n-1} 2\,x^i = \frac{x^{n+1} + x^n - x - 1}{x-1}. \tag{16}$$

Combining equations 13, 14, 16, and 5 finally results in

$$\mathcal{V}_3(x) = \frac{n\,x^{n+2} - 2\,x^{n+1} - n\,x^n + 2\,x}{(x-1)^2}. \tag{17}$$

3.2 Convolution

In the composition technique, we composed the set **K** of cell pairs out of subsets. In the convolution technique, the paths between the pairs are composed of smaller paths. The idea behind this is that the symmetry in the physical architecture generally allows us to find a small set of paths that form a "base" for the original set of paths. Each original path can then be composed by a choice of a single path out of each "base set". The length distribution is then given by the convolution of the distributions for these base sets. A known

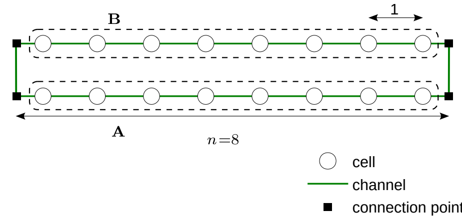

Figure 4: Line of n cells connected at sides to another line of n cells.

[1]The extraction of the distribution from the polynomial will be the subject of section 4.

property of generating polynomials (or Z-transforms in general) is that the generating polynomial of a convolution of two distributions equals the product of the generating polynomials of each of the distributions.

Again consider the example of figure 3. This figure shows that paths from cells in **A** to cells in **B** can be composed of a path from **A** to **C**, a path from **C** to **D** and a path from **D** to **B**. Let us denote the sets as

$$\mathbf{K} = \mathbf{A} \times \mathbf{B}, \qquad \mathbf{K}_1 = \mathbf{A} \times \mathbf{C}$$
$$\mathbf{K}_2 = \mathbf{C} \times \mathbf{D}, \ \text{and} \ \mathbf{K}_3 = \mathbf{D} \times \mathbf{B} \tag{18}$$

and their corresponding length distributions as $f(\ell)$, $f_1(\ell)$, $f_2(\ell)$, and $f_3(\ell)$ and the generating polynomials as $\mathcal{V}_4(x)$, $\mathcal{V}_{4,1}(x)$, $\mathcal{V}_{4,2}(x)$, and $\mathcal{V}_{4,3}(x)$. The composition of paths results in a convolution of length distributions and hence a multiplication of generating polynomials

$$f(\ell) = f_1(\ell) * f_2(\ell) * f_3(\ell) \tag{19}$$
$$\mathcal{V}_4(x) = \mathcal{V}_{4,1}(x)\,\mathcal{V}_{4,2}(x)\,\mathcal{V}_{4,3}(x). \tag{20}$$

Note that $f_1(\ell)$ and $f_3(\ell)$ are equal ($\mathcal{V}_{4,1}(x) = \mathcal{V}_{4,3}(x)$). The generating polynomial for the set $\mathbf{K}_1$ is easily obtained as (number the cells from right to left in **A**)

$$\mathcal{V}_{4,1}(x) = \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}. \tag{21}$$

Since there is only one pair in $\mathbf{K}_2$ (with length 1), the corresponding generating polynomial is simply $\mathcal{V}_{4,2}(x) = x$ and equation 20 results in

$$\mathcal{V}_4(x) = \frac{x\,(x^n - 1)^2}{(x-1)^2} \tag{22}$$

which is, of course, equal to $\mathcal{V}_2(x)$ (equation 10).[2]

The convolution technique can be used whenever the paths can be split into a set of independent parts. The length distributions (or better generating polynomials) can then be calculated separately (which results in easier problems) and the convolution of these distributions (multiplication of the polynomials) results in the final distribution (polynomial). It is clear that the use of generating polynomials greatly reduces the calculation overhead in the convolution technique as compared to the use of the distributions themselves.

4. EXTRACTION OF DISTRIBUTIONS

The short notation of the generating polynomial as a rational function is very beneficial for obtaining the generating polynomials for multi-dimensional and possibly complex architectures (multiplying such polynomials is easy). Of course, the drawback is that we have to use a polynomial division if we want to recover the analytical form of the distribution again. In this section, we present a method to extract the distributions from generating polynomials. Due to the specific form of our generating polynomials, the calculation is much simpler than traditional (more general) methods such as partial ratios [9] or calculating an inverse Z-transform [22].

Because the unit distance in all our calculations is 1, the generating polynomials are rational functions with a denominator $(x-1)^i$.[3] The reason is that they are composed of several geometric series with factor x:

$$\sum_{i=a}^{b} c\,x^i = c\,\frac{x^{b+1} - x^a}{x - 1}. \tag{23}$$

Hence, each generating polynomial can be written as a sum of terms $x^n/(x-1)^i$.

To extract the distribution from its generating polynomial, we introduce the following theorem [12]:

Theorem *A term $x^n/(x-1)^i$ can be written as*

$$\frac{x^n}{(x-1)^i} = \sum_{\ell=0}^{n-i} \binom{n-\ell-1}{i-1} x^\ell + \frac{O\left(x^{i-1}\right)}{(x-1)^i}, \tag{24}$$

with

$$\binom{n-\ell-1}{i-1} = \frac{1}{(i-1)!} \prod_{j=1}^{i-1} (n-\ell-j). \tag{25}$$

The proof of the theorem is presented in the appendix. The first term in the right hand side of equation 24 is the *quotient* term, the second one the *remainder* term, after applying the polynomial division. The remainder term will be of no further use for our calculations since the sum of all remainder terms of a generating polynomial is always 0. Indeed, the denominator of the generating polynomial is a divisor polynomial of the generating polynomial's entire numerator. We therefore do not need to detail the remainder term.

Consequence *Since the combinatorial function in the quotient term is zero for $\ell = n-1, n-2, \ldots, n-i+1$ (equation 25), the upper summation index in equation 24 can be chosen from $n-i$ to $n-1$ without changing the result.*

This proves to be very helpful in simplifying the expressions. Consider a generating polynomial

$$\mathcal{V}(x) = \frac{\sum_{j=0}^{k} a_j\, x^{b_j}}{(x-1)^i}, \tag{26}$$

with $a_j \in \mathbb{R}$ and $b_j \in \mathbb{N}^+$. Application of equation 24 on all terms results in

$$\mathcal{V}(x) = \sum_{j=0}^{k} a_j \sum_{\ell=0}^{b_j - i} \binom{b_j - \ell - 1}{i-1} x^\ell \tag{27}$$

since all remainder terms vanish in the final solution.

The final distribution is obtained as

$$f(\ell) = \sum_{j=0}^{k} a_j \left.\binom{b_j - \ell - 1}{i-1}\right|_{0 \le \ell \le b_j - i}. \tag{28}$$

Note that the different summation bounds in equation 27 result in different ranges for ℓ in the distribution support. The number of ranges can be lowered by using the degree of freedom to choose the best summation upper bound (from $b_j - i$ to $b_j - 1$). These ranges thus are implicitly present in the representation of the distribution by its generating polynomial. Being able to delay the calculations for different ranges (each with a separate result) to the very last computation step, is an important benefit of generating polynomials.

[2]Note that we used the composition of the paths as described here to write the summation in equation 10 in the first place. Although this summation was as straightforward as the multiplication of the generating polynomials, this is no longer the case for more complicated examples.

[3]The extraction technique is also valid for architectures where the unit distance for all or some of the channels differs from 1 (the equations can be a little more complex). A more general extraction technique is presented in [20].

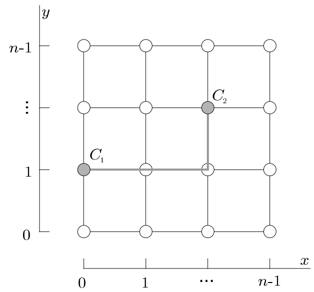

Figure 5: Square Manhattan grid with side n.

5. EXAMPLES

5.1 Two-dimensional isotropic grid

As an example of the ease of interconnection length calculations when using generating polynomials, let us consider the two-dimensional isotropic square Manhattan grid used in Davis' model [4] (and many others) (figure 5). Because of the square symmetry, we can compose each path between two cells C_1 and C_2 from its x and its y components and use the convolution technique to find the entire length distribution. Moreover, the distributions for the x and y components are equal and are the solution of the problem presented in figure 2. The corresponding generating polynomial is $\mathcal{V}_1(x)$ (equation 5). We only have to take into account the fact that the connections from cells to themselves should not be counted (subtract $n^2\,x^0$) and that the convolution will count a connection from C_1 to C_2 twice (also as a connection from C_2 to C_1). The generating polynomial of the distribution for the two-dimensional grid is therefore

$$\mathcal{V}_5(x) = \frac{(\mathcal{V}_1(x))^2 - n^2}{2} \tag{29}$$

$$= \frac{2x^{2n+2} - 2nx^{n+3} - 4x^{n+2} + 2nx^{n+1} + \mathrm{O}\left(x^3\right)}{(x-1)^4} \tag{30}$$

Using equations 26 and 28 results in (with obvious choices for ranges)

$$f_5(l) = 2\,\binom{2\,n-\ell+1}{3}\Bigg|_{0\le\ell\le 2\,n} - 2\,n\,\binom{n-\ell+2}{3}\Bigg|_{0\le\ell\le n} - 4\,\binom{n-\ell+1}{3}\Bigg|_{0\le\ell\le n} + 2\,n\,\binom{n-\ell}{3}\Bigg|_{0\le\ell\le n}$$

$$= \begin{cases} 2\binom{2n-\ell+1}{3} - 2n\binom{n-\ell+2}{3} - 4\binom{n-\ell+1}{3} + 2n\binom{n-\ell}{3} & (0\le\ell\le n) \\ 2\binom{2n-\ell+1}{3} & (n<\ell\le 2n) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{\ell}{3}\left(6\,n^2 - 6\,\ell\,n + \ell^2 - 1\right) & (0\le\ell\le n) \\ \frac{(2\,n-\ell-1)\,(2\,n-\ell)\,(2\,n-\ell+1)}{3} & (n<\ell\le 2\,n) \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

which is the same distribution as found in [4] (equation B7). The average length can be directly found from the generating polynomial (and is easily obtained by using equations 29, 7 and 9)

$$\ell_{avg} = \left(\frac{d\mathcal{V}_5(x)}{dx}\,\frac{1}{\mathcal{V}_5(x)}\right)\Bigg|_{x=1} = \left(\frac{d\mathcal{V}_1(x)}{dx}\,\mathcal{V}_1(x)\,\frac{2}{(\mathcal{V}_1(x))^2 - n^2}\right)\Bigg|_{x=1} = \frac{(n^2-1)}{3\,n}\,\frac{2\,n^4}{n^4-n^2} = \frac{2\,n}{3}. \tag{32}$$

A comparison of the above calculations (a simple summation to obtain $\mathcal{V}_1(x)$, straightforward multiplication of generating polynomials and an extraction process that reduces to the summation of the right set of combinatorial functions) with the more difficult enumeration that was used in [4] (enumerating all distributions for each single cell – difficult to calculate – over all cells), one truly appreciates the generating polynomial representation.

5.2 More complicated architectures

Where the traditional enumeration techniques lead to intractable calculation difficulties or time-consuming and error-prone computations, the technique of generating polynomials, together with the use of composition and convolution rules, results in an efficient way to obtain the length distributions, even for complicated architectures.

Consider, for example, the anisotropic grid of figure 1. Assume that the special connections between **A** and **B** (between two connection points) have length k. We make the following observations (due to symmetry, figure 1):

1. Paths that start from **C** have exactly one symmetrical counterpart (with equal length) that starts from **D**. Hence the total set **K** of paths from **A** to **B** is two times the set $\mathbf{K}'$ of paths from **C** to **B**.
2. The set $\mathbf{K}'$ can be composed of the set $\mathbf{K}''$ of paths from **C** to **E** and the set $\mathbf{K}'''$ of paths from **C** to **F**.
3. All paths in $\mathbf{K}''$ have to pass either the top or the middle connection points between **A** and **B**.
4. All paths in $\mathbf{K}'''$ have to pass the middle connection points between **A** and **B**.

The third observation leads us to the conclusion that all paths, for both the parts in **C** and **E**, have to travel the distance to the cells closest to the other set of cells as in figure 3 (but without the connection between **C** and **D** of figure 3; x component of the path is represented by $\mathcal{V}_{4,1}(x)$ in both **C** and **E**) and that the y component is represented by the ring structure of figure 4 (the generating polynomial equals $\mathcal{V}_3(x)$). Together with the special connection that takes length k, the generating polynomial for the set $\mathbf{K}''$ can be written as

$$\mathcal{V}_6''(x) = \mathcal{V}_{4,1}(x)\,\mathcal{V}_3(x)\,x^k\,\mathcal{V}_{4,1}(x) = \frac{(x^n-1)^2 x^k\,\left(nx^{n+2} - 2x^{n+1} - nx^n + 2x\right)}{(x-1)^4}. \tag{33}$$

Similarly, the fourth observation requires all paths to move to the corner element ($\mathcal{V}_{4,1}^2(x)$ in both **C** and **F**) and cross the special connection (length $k+1$!), which results in

$$\mathcal{V}_6'''(x) = \mathcal{V}_{4,1}^2(x)\,x^{k+1}\,\mathcal{V}_{4,1}^2(x) = \frac{(x^n-1)^4\,x^{k+1}}{(x-1)^4}. \tag{34}$$

Observations 1 and 2 then lead to the final polynomial

$$\mathcal{V}_6(x) = 2\,\left(\mathcal{V}_6''(x) + \mathcal{V}_6'''(x)\right)$$

ℓ	0	1	2	3	4	5	6	>6
No.	0	0	6	20	24	12	2	0

Table 1: Length distribution for figure 1 for $k = 1$ and $n = 2$.

$$= 2\,\frac{(x^n-1)^2\,x^k}{(x-1)^4}\,\left(x^{2\,n+1} + n\,x^{n+2} - 4\,x^{n+1} - n\,x^n + 3\,x\right).$$

This equation can be simplified as

$$\mathcal{V}_6(x) = 2\,G(3) - 4\,G(2) + 2\,G(1) + 2\,\frac{x^{4\,n+k+1} - 4\,x^{n+k+1} + 3\,x^{k+1}}{(x-1)^4}, \qquad (35)$$

with

$$G(h) = \frac{n\,x^{h\,n+k+2} - 6\,x^{h\,n+k+1} - n\,x^{h\,n+k}}{(x-1)^4}. \qquad (36)$$

Using equations 26 and 28 we find the corresponding distributions (omitting the remainder terms)

$$g(h,\ell) = \left(n\binom{h\,n+k-\ell+1}{3} - 6\binom{h\,n+k-\ell}{3} - n\binom{h\,n+k-\ell-1}{3}\right)\Bigg|_{0\le\ell\le h\,n+k-1}$$
$$= \left((h\,n+k-\ell-1)\,\left(n\,(h\,n+k-\ell-1) - (h\,n+k-\ell)\,(h\,n+k-\ell-2)\right)\right)\big|_{0\le\ell\le h\,n+k-1}$$

$$f_6(\ell) = \begin{cases} f_{6,4}(\ell) = 2\binom{4\,n+k-\ell}{3} \\ f_{6,3}(\ell) = f_{6,4}(\ell) + 2\,g(3,\ell) \\ f_{6,2}(\ell) = f_{6,3}(\ell) - 4\,g(2,\ell) \\ f_{6,1}(\ell) = f_{6,2}(\ell) + 2\,g(1,\ell) - 8\binom{n+k-\ell}{3} \\ f_{6,0}(\ell) = f_{6,1}(\ell) + (k-\ell)\,(k-\ell-1)\,(k-\ell-2) \end{cases}$$

where $f_{6,h}(\ell), h > 0$ is valid for $((h-1)\,n + k \le \ell \le h\,n + k - 1)$, and $f_{6,0}(\ell)$ for $(0 \le \ell \le k-1)$ and where $f_6(\ell) = 0$ for all other values of ℓ. The final equations for the distribution $f_6(\ell)$ are presented in equation 37. One can check the distribution for some simple cases, by counting the number of cells in $\mathbf{B}$ at a distance ℓ from each cell in $\mathbf{A}$. The result for $k = 1$ and $n = 2$ is presented in table 1. One can verify that these values are also obtained by substituting ℓ, k and n by their appropriate values in equation 37.

The same technique can also be applied to compute the length distribution of the connections from cells to the I/O-pads [17]. The automation of constructing length distributions for any physical architecture comes to mind. However, the intelligence needed to make efficient use of symmetry (currently) prevent an automated version of the technique. The extraction, on the other hand, (the most difficult part in terms of computation) can be easily automated using symbolic calculator tools. This is indeed an important advancement with respect to the current situation.

6. CONCLUSION

Researchers in the field of interconnection length estimation have been and will be confronted with the problem of enumerating all distances (computed along the shortest path) between cells in a physical architecture. In this paper, we have shown that the enumeration of wire length distributions can be done much more efficiently by representing them as a generating polynomial. The construction of the generating polynomial is facilitated by using the inherent symmetry in the physical architecture. Two techniques have been described in this paper: composition and convolution. Both techniques reduce the construction of a generating polynomial to a very simple problem.

The extraction of the length distribution from its generating polynomial is generally the most difficult step. However, the specific form of generating polynomials that represent length distributions reduces the problem to a substitution of terms in the polynomial by summations of combinatorial functions with only a few factors. These can easily be simplified (by symbolic calculator tools) and the extraction process automated to immediately lead to the solution.

Generating polynomials are beneficial as a representation for length distributions because they (i) make it easier to exploit the inherent symmetry in the problem, (ii) provide a very compact representation of the distribution, (iii) facilitate the construction of the length distribution by easily enabling composition and convolution techniques, and (iv) implicitly contain the different length ranges for which the distribution has different function representations. With generating polynomials, the computation of length distributions in complex physical architectures comes within reach, as is shown in the example of figure 1.

7. REFERENCES

[1] H. B. Bakoglu. *Circuits, Interconnections, and Packaging for VLSI.* Addison-Wesley, 1990.

[2] A. Caldwell, A. B. Kahng, F. Koushanfar, H. Lu, I. Markov, M. Oliver, and D. Stroobandt. GTX: The MARCO GSRC technology extrapolation system. To be published in Proc. ACM/IEEE Design Automation Conf. `http://vlsicad.cs.ucla.edu/GSRC/GTX/`, 2000.

[3] P. Christie and D. Stroobandt. The interpretation and application of Rent's rule. *IEEE Trans. on VLSI Systems, Special Issue on System-Level Interconnect Prediction*, September 2000.

[4] J. A. Davis, V. K. De, and J. D. Meindl. A stochastic wire-length distribution for gigascale integration (GSI) – PART I: Derivation and validation. *IEEE Trans. on Electron Devices*, 45 (3): pages 580–589, March 1998.

[5] W. E. Donath. Placement and average interconnection lengths of computer logic. *IEEE Trans. Circuits & Syst.*, vol. CAS–26: pages 272–277, 1979.

[6] J. C. Eble, V. K. De, D. S. Wills, and J. D. Meindl. A generic system simulator (GENESYS) for ASIC technology and architecture beyond 2001. In *Proc. 9th Annual IEEE Intl. ASIC Conf.*, pages 193–196, 1996.

[7] B. M. Geuskens. Modeling the influence of multilevel interconnect on chip performance, 1997. *Ph.D. thesis*, Rensselaer Polytechnic Institute, Troy, New York, `http://latte.cie.rpi.edu/ripe.html`.

[8] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Academic Press, 1980. Corrected and enlarged edition.

[9] R. P. Grimaldi. *Discrete and Combinatorial Mathematics.* Addison-Wesley Publishing Co., 1986.

$$
f_6(\ell) = \begin{cases} 0 & (0 \le \ell \le k-1) \\ (\ell-k+2)\,(\ell-k+1)\,(\ell-k) & (k \le \ell \le n+k-1) \\ -\frac{2}{3}\,(n+k-l-1)\,(3\,n\,(n+k-l-1)-5\,(n+k-l)\,(n+k-l-2)) & \\ \quad +(\ell-k+2)\,(\ell-k+1)\,(\ell-k) & (n+k \le \ell \le 2\,n+k-1) \\ 2\,(3\,n+k-l-1)\,(n\,(3\,n+k-l-1)-(3\,n+k-l)\,(3\,n+k-l-2)) & \\ \quad +\frac{1}{3}\,(4\,n+k-l)\,(4\,n+k-l-1)\,(4\,n+k-l-2) & (2\,n+k \le \ell \le 3\,n+k-1) \\ \frac{1}{3}\,(4\,n+k-\ell)\,(4\,n+k-\ell-1)\,(4\,n+k-\ell-2) & (3\,n+k \le \ell \le 4\,n+k-1) \\ 0 & \text{otherwise.} \end{cases} \tag{37}
$$

[10] B. S. Landman and R. L. Russo. On a pin versus block relationship for partitions of logic graphs. *IEEE Trans. on Comput.*, vol. C–20: pages 1469–1479, 1971.

[11] H. M. Ozaktas and J. Goodman. Implications of interconnection theory for optical digital computing. *Applied Optics*, vol. 31: pages 5559–5567, 1992.

[12] D. Stroobandt. Analytical methods for a priori wire length estimates in computer systems, 1998. Ph.D. thesis (translated from Dutch), Ghent University.

[13] D. Stroobandt. Tutorial: A priori wire length estimates based on Rent's rule. In *Workshop on System-Level Interconnect Prediction*, pages 3–50, 1999.

[14] D. Stroobandt and J. Van Campenhout. Estimating interconnection lengths in three-dimensional computer systems. *IEICE Trans. on Inf. & Syst., Special Issue on Synthesis and Verification of Hardware Design*, vol. E80–D (no. 10): pages 1024–1031, October 1997.

[15] D. Stroobandt and J. Van Campenhout. Accurate interconnection length estimations for predictions early in the design cycle. VLSI Design, Special Issue on Physical Design in Deep Submicron, vol. 10, 1999.

[16] D. Stroobandt, H. Van Marck, and J. Van Campenhout. On the use of generating polynomials for the representation of interconnection length distributions. In *Proc. 4th Intl. Workshop on Symbolic Methods and Applications in Circuit Design (SMACD '96)*, pages 70–73. Katholieke Universiteit Leuven, October 1996.

[17] D. Stroobandt, H. Van Marck, and J. Van Campenhout. Estimating logic cell to I/O pad lengths in computer systems. In *Proc. Workshop on Synthesis and System Integration of Mixed Technologies*, pages 192–198. S. Insatsu, Osaka, Japan, December 1997.

[18] D. Sylvester and K. Keutzer. System-level performance modeling with BACPAC – Berkeley advanced chip performance calculator. In *Workshop on System- Level Interconnect Prediction*, pages 109–114, 1999.

[19] J. Van Campenhout, H. Van Marck, J. Depreitere, and J. Dambre. Optoelectronic FPGAs. *IEEE Journal of Selected Topics in Quantum Electronics on Smart Photonic Components, Interconnects, and Processing*, vol. 5 (no. 2): pages 306–315, March/April 1999.

[20] H. Van Marck. Kwantitatieve studie en modellering van opto-elektronische driedimensionale interconnectiestructuren, 2000. Ph.D. dissertation (in Dutch), Ghent University, Belgium.

[21] H. Van Marck and J. Van Campenhout. Modeling and evaluating optoelectronic architectures. In *Optoelectronics II*, volume 2153 of *SPIE Proc. Series*, pages 307–314, SPIE, 1994.

[22] R. Vich. *Z Transform Theory and Applications.* D. Reidel Publishing Company, 1987.

APPENDIX

The proof of equation 24 goes by induction on i [12].

1. For $i = 1$, equation 24 reduces to

$$\frac{x^n}{x-1} = \sum_{\ell=0}^{n-1} x^\ell + \frac{O(1)}{x-1} \tag{38}$$

and is proven because

$$\sum_{\ell=0}^{n-1} x^\ell = \frac{x^n-1}{x-1} = \frac{x^n}{x-1} - \frac{1}{x-1}. \tag{39}$$

2. If equation 24 is valid for $i = j \ge 1$, then it follows that

$$\frac{x^n}{(x-1)^{j+1}} = \frac{1}{(x-1)}\,\frac{x^n}{(x-1)^j} = \sum_{m=0}^{n-j} \binom{n-m-1}{j-1} \frac{x^m}{x-1} + \frac{O\left(x^{j-1}\right)}{(x-1)^{j+1}},$$

and, using equation 39,

$$\frac{x^n}{(x-1)^{j+1}} = \sum_{m=1}^{n-j} \binom{n-m-1}{j-1} \left(\sum_{\ell=0}^{m-1} x^\ell + \frac{1}{x-1}\right) + \binom{n-1}{j-1} \frac{1}{x-1} + \frac{O\left(x^{j-1}\right)}{(x-1)^{j+1}}.$$

Changing the summation indices results in

$$\frac{x^n}{(x-1)^{j+1}} = \sum_{\ell=0}^{n-j-1} \left(x^\ell \sum_{m=\ell+1}^{n-j} \binom{n-m-1}{j-1} \right) + \frac{O\left(x^j\right)}{(x-1)^{j+1}}.$$

The sum of combinatorial functions can be calculated by substituting the summation index by $m' = n - m - j$:

$$\sum_{m=\ell+1}^{n-j} \binom{n-m-1}{j-1} = \sum_{m'=0}^{n-\ell-j-1} \binom{m'+j-1}{j-1}. \tag{40}$$

In [8], *we find that the following equation holds*

$$\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}. \tag{41}$$

Using equation 41 with k substituted by m', n by $j-1$ and m by $n-\ell-j-1$, results in

$$\sum_{m'=0}^{n-\ell-j-1} \binom{m'+j-1}{j-1} = \binom{n-\ell-1}{j}. \tag{42}$$

Finally, this leads to

$$\frac{x^n}{(x-1)^{j+1}} = \sum_{\ell=0}^{n-(j+1)} \binom{n-\ell-1}{(j+1)-1} x^\ell + \frac{O\left(x^j\right)}{(x-1)^{j+1}},$$

which proves that equation 24 is also valid for $i = j + 1$.