On the Use of Generating Polynomials for the Representation of Interconnection Length Distributions

Dirk Stroobandt* Herwig Van Marck† Jan Van Campenhout
Department of Electronics and Information Systems
University of Ghent
Sint-Pietersnieuwstraat 41
B-9000 Gent, Belgium
{dstr, hvm, jvc}@elis.rug.ac.be

Theoretical estimates of interconnection lengths in computer systems are important for various reasons. First of all, placement and routing tools use such estimates to limit the search in the solution space. Secondly, the estimates can be used to compare different computer architectures before they are actually built. Our work on interconnection length estimations [1, 2] is based on the work of Donath [3] and on Rent's rule [4, 5]. To obtain the theoretical results, the distribution of the number of interconnections with respect to their length, must be computed. In this paper we will focus on the enumeration problem that is part of this computation.

We will formulate the problem by considering the example of figure 1. This figure shows a stack of two-dimensional Manhattan grids of size n. The two Manhattan planes are only interconnected to each other at the corners of the grid (anisotropic grid). We connect two nodes N_1 (top plane) and N_2 (bottom plane) along the shortest path. The *distance* between nodes N_1 and N_2 is the length of this shortest path. We want to obtain the (discrete) distribution $f(l)$ of the number of possible node pairs (N_1,N_2) at distance l. A first problem that can be observed, is the fact that we have to choose the shortest path out of four possible paths (figure 1). Since we want to obtain an analytical expression without binding n to a specific value, a simple exhaustive enumeration technique is very cumbersome.

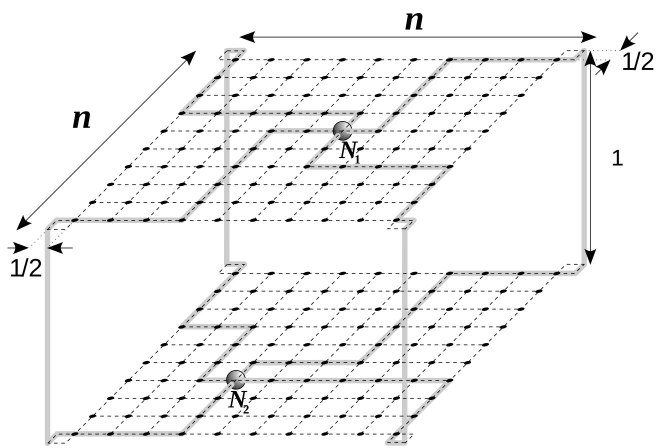

Figure 1: The distance between two nodes N_1 and N_2 in the example grid.

A more elegant way to compute the distribution $f(l)$ can be found by introducing another representation of this distribution. A discrete distribution can be represented by its moment-generating polynomial function (further referred to as the *generating polynomial* for the distribution)

$$P(x) = \sum_{l=0}^{\infty} f(l)\, x^l. \qquad (1)$$

Since the distance in a real computer architecture always is finite (depending on the size n of the architecture), there always exists a k for which $f(l) = 0$ for all $l > k$.

The generating polynomial $P(x)$ not only contains all information about the distribution itself,

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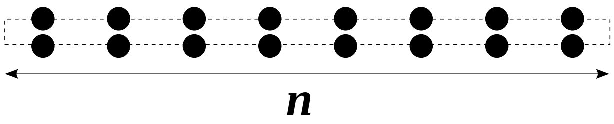

Figure 2: A set of n nodes interconnected to another set of n nodes.

but it also provides an easy way to compute the average value of the distribution and all higher order moments. This is done by differentiating $P(x)$ as many times as the order of the moment we want to compute and evaluating the result for $x = 1$. So, the total number of interconnections is given by:

$$\lim_{x\to 1} P(x)$$

and the average interconnection length can be found as

$$\lim_{x\to 1} \frac{\frac{dP(x)}{dx}}{P(x)}.$$

Another property of generating polynomials is that the generating polynomial of a convolution of two distributions equals the product of the generating polynomials of each of the distributions

$$\left.\begin{array}{l} P_a(x) = \sum_{l=0}^{\infty} f_a(l)\, x^l \\ P_b(x) = \sum_{l=0}^{\infty} f_b(l)\, x^l \\ P(x) = \sum_{l=0}^{\infty} (f_a * f_b)(l)\, x^l \end{array}\right\} \Rightarrow P(x) = P_a(x)\, P_b(x).$$

As an example of the use of generating polynomials, one can calculate the generating polynomial representing the interconnection length distribution for the grid shown in figure 2. In this grid, a set of n nodes (at the bottom) is fully interconnected with another set of n nodes (at the top) using the channels shown in figure 2. The generating polynomial is calculated as

$$\begin{aligned} P_1(x) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} x^{\min(i+j+1,\, 2n-i-j-1)} && (2) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1} x^{i+j+1} + \sum_{i=0}^{n-1}\sum_{j=n-i}^{n-1} x^{2n-i-j-1} \\ &= \frac{n\,x^{n+2} - 2\,x^{n+1} - n\,x^n + 2\,x}{(x-1)^2}. && (3) \end{aligned}$$

Let us now return to the original example of figure 1 and let us try to calculate the generating polynomial for that grid. We could enumerate all four possible paths between all node combinations and eliminate all but the shortest path. This leads to an equation similar to equation 2 but with four summation variables (two for locating one point in the grid, the other two for locating the other point) and with four arguments to the minimum-function (for the four possible paths). This is quite a difficult equation to solve. However, the technique of the generating polynomials enables us to calculate this far more efficiently, as will be shown next.

Due to the additive nature of the distance in Manhattan grids it is often possible to express the interconnection length distribution as a convolution of other simpler distributions. This leads to the product of two generating polynomials. We can see that the example of figure 1 is a two-dimensional equivalent of figure 2 thus resulting in

$$P_2(x) = P_1(x)\, P_1(x)\, x$$

where $P_1(x)$ is the generating polynomial representing the interconnection length distribution for the situation presented in figure 2 (equation 3). The multiplication by x represents the inclusion of the part of the path between the two planes (at distance 1). This way, the generating polynomial is found to be

$$P_2(x) = \frac{x\,\left(n\,x^{n+2} - 2\,x^{n+1} - n\,x^n + 2\,x\right)^2}{(x-1)^4}. \quad (4)$$

At this moment, the generating polynomial is represented by a rational function. Note that the denominator of this rational function is always a divisor polynomial of the numerator. We are already able to find the total number of interconnections and the average interconnection length without even having to calculate the interconnection length distribution itself. However, if we want a closed form for this distribution, we need to convert the rational function back to a polynomial. Generally, this conversion can be done by using the inverse $\mathcal{Z}$-transform

$$\begin{aligned} f(-l) &= \mathcal{Z}^{-1}(P_2(x)) \\ &= \sum_{i=1}^{K} \operatorname*{res}_{x=x_i} P_2(x)\; x^{l-1} \end{aligned}$$

where the residue is given by

$$\operatorname*{res}_{x=x_i} P_2(x)\, x^{l-1} = \frac{1}{(m-1)!} \lim_{x\to x_i} \frac{d^{m-1}}{dx^{m-1}} \left[(x-x_i)^m P_2(x)\, x^{l-1} \right].$$

Because our grids have spacing one, it is not unusual that the denominator of the rational function contains only a power of $(x-1)$. Therefore we can simplify the calculation of residues by using the following equation

$$\frac{x^n}{(x-1)^k} = \sum_{l=0}^{n-k} \frac{(n-l-1)!}{(k-1)!\,(n-l-k)!}\, x^l \qquad (5)$$

$$+ \frac{\sum_{l=0}^{k-1} \frac{n!\,(-1)^{k-1-l}}{(n-k)!\,(n-l)!\,l!\,(k-1-l)!}\, x^l}{(x-1)^k} \qquad (6)$$

The first terms in this equation (equation part 5) will be called the *quotient terms*, the others (equation part 6) the *rest terms*. This equation can be proven by using the $\mathcal{Z}$-transform.

By expanding the numerator of $P_2(x)$ (equation 4) and using the quotient terms in equation 5, one finds the full representation of the generating polynomial after combining all terms with equal l. Note that in the expansion of the numerator, the sums of all rest terms must equal zero since the denominator is always a divisor polynomial of the numerator in the closed form of the generating polynomial. Therefore there is no need to include the rest terms in the calculation.

The summation boundaries differ for all terms in the expansion (although some boundaries can be combined since the quotient terms are zero for $n-1 \le l \le n-k+1$) and this leads to a different interconnection length distribution in different intervals of the distribution support. The interconnection length distribution represented by $P_2(x)$ can be found to be

$$f(l) = \begin{cases} \frac{2l^3-6l^2+4l}{3} & 0 \le l \le n \\ -\frac{8n^3+12(1-l)n^2+4n+2l\left(l^2-3l+2\right)}{3} & n < l \le 2n \\ n^2 & l = 2n+1 \end{cases}$$

As a conclusion we can state that the calculation of interconnection length distributions is much easier when using generating polynomials. These generating polynomials provide a compact representation and allow immediate calculation of the average interconnection length. Furthermore, the different intervals of the distribution support are implicitly present in the generating polynomial representation. This means that these intervals do not occur explicitly except at the very last phase of computation. Postponing the division of the distribution support into parts significantly reduces the total number of computations.

Further research will focus on automatic generation of the generating polynomials for generic architectures.

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